To appear in the Notre Dame
Journal of Formal Logic

Ekman's Paradox

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Abstract

Prawitz observed that Russell's paradox in naive set theory yields a derivation of absurdity whose reduction sequence loops. Building on this observation, and based on numerous examples, Tennant claimed that this looping feature, or more general, the fact that derivations of absurdity do not normalize, is characteristic of the paradoxes. Striking results by Ekman show that looping reduction sequences are already obtained in minimal propositional logic, when certain reduction steps, which are prima facie plausible, are considered in addition to the standard ones. This shows that the notion of reduction is in need of clarification. Referring to the notion of identity of proofs in general proof theory, we argue that reduction steps should not merely remove redundancies, but must respect the identity of proofs. Consequentially, we propose to modify Tennant's paradoxicality test by basing it on this refined notion of reduction.

Keywords: Paradoxes, Russell's paradox, Normalization, General Proof Theory, Identity of Proofs
Mathematics Subject Classification: 03A05, 00A30, 03F05, 03F07

Tennant (1982; 1995), building on ideas of Prawitz (1965), proposed the following proof-theoretic test for paradoxicality: A derivation of absurdity is paradoxical, whenever each reduction sequence starting from this derivation eventually loops. Results by Ekman (1994; 1998) show that already minimal propositional logic gives rise to looping reduction sequences, if the standard reductions are extended with new reductions that remove redundancies in a way which is prima facie very plausible. Based on the discussion on the identity of proofs within general proof theory, we argue that the characterization of reductions as transformations of derivations that remove redundancies, is too broad. It needs to be augmented by the feature that the addition of new reduction steps must preserve the identity of proofs established by the standard reductions.

This is not the case with the new reductions proposed by Ekman. We conclude that Tennant's test is too coarse, and we suggest how to improve it by adopting the more restricted notion of reduction.

1 The Prawitz-Tennant Analysis of Paradoxes

The natural deduction system for minimal implicational logic consists of the following introduction and elimination rules for implication as its only primitive rules of inference:

$$\dfrac{\begin{array}{c}[A]\\ B\end{array}}{A \to B}\,(\to\mathrm{I}) \qquad \dfrac{A \to B \quad A}{B}\,(\to\mathrm{E})$$

Negation is defined as implication of absurdity $\bot$, i.e., $\neg A =_{\mathrm{def}} A \to \bot$.

In natural deduction systems, the application in a derivation of an introduction rule followed immediately by an application of the corresponding elimination rule constitutes a redundancy. Redundancies can be eliminated by rearranging the structure of derivations according to certain reductions. The reduction for implication is the following:

$$\dfrac{\dfrac{\begin{array}{c}\overset{1}{[A]}\\ \mathcal{D}\\ B\end{array}}{A \to B}\,(\to\mathrm{I})(1) \qquad \begin{array}{c}\mathcal{D}'\\ A\end{array}}{B}\,(\to\mathrm{E}) \qquad \rhd_{\to} \qquad \begin{array}{c}\mathcal{D}'\\ {[A]}\\ \mathcal{D}\\ B\end{array}$$

The occurrence of $A \to B$ in the left derivation, which is removed by this reduction step, will be called a *redundant implication formula*. A derivation is normal if and only if it is redundancy-free. In his book on natural deduction, Prawitz (1965) showed that all derivations in minimal (as well as intuitionistic and classical) logic can be transformed into normal form.

In Appendix B to this book, Prawitz considered a system for naive set theory obtained by extending the one for minimal logic with an introduction and an elimination rule for formulas of the form $t \in \{x : A\}$ to express set-theoretical comprehension:

$$\dfrac{A[t/x]}{t \in \{x : A\}}(\in\mathrm{I}) \qquad \dfrac{t \in \{x : A\}}{A[x/t]}(\in\mathrm{E})$$

where $A[t/x]$ is the result of replacing t for x in A. Also in this case an application of the introduction rule immediately followed by an application of the corresponding elimination rules constitutes a redundancy which can be eliminated according to the following $\in$-*reduction*:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A[t/x]\end{array}}{t \in \{x : A\}}}{A[t/x]} \qquad \rhd_{\in} \qquad \begin{array}{c}\mathcal{D}\\ A[t/x]\end{array} \tag{1}$$

The occurrence of $t \in \{x : A\}$, which is removed by this reduction, will be called a *redundant* $\in$*-formula*.

Taking λ to be $r \in r$, where r is the Russell term $\{x : x \notin x\}$, an application of the $\in$-introduction rule allows one to pass over from $\neg\lambda$ to λ, and an application of the $\in$-elimination rule from λ back to $\neg\lambda$. This yields Russell's paradox in the form of the following derivation of absurdity $\bot$ in minimal logic extended with $(\in \mathrm{I})$ and $(\in \mathrm{E})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}(\in \mathrm{E}) \quad \overset{1}{\lambda}}{\bot}(\to \mathrm{E})}{\neg\lambda}(\to \mathrm{I})(1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}(\in \mathrm{E}) \quad \overset{1}{\lambda}}{\bot}(\to \mathrm{E})}{\neg\lambda}(\to \mathrm{I})(1)}{\lambda}(\in \mathrm{I})}{\bot}(\to \mathrm{E}) \tag{2}$$

Since the encircled occurrence of $\neg\lambda$ is a redundant implication formula, this derivation is not normal. By applying implication reduction $\rhd_{\to}$ we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}(\in \mathrm{E}) \quad \overset{1}{\lambda}}{\bot}(\to \mathrm{E})}{\neg\lambda}(\to \mathrm{I})(1)}{\lambda}(\in \mathrm{I})}{\neg\lambda}(\in \mathrm{E}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}(\in \mathrm{E}) \quad \overset{1}{\lambda}}{\bot}(\to \mathrm{E})}{\neg\lambda}(\to \mathrm{I})(1)}{\lambda}(\in \mathrm{I})}{\bot}(\to \mathrm{E})$$

Here the encircled occurrence of λ is a redundant $\in$-formula. By applying the $\in$-reduction $\rhd_{\in}$ we obtain the derivation (2) we started with. Since at each of the two steps there was only a single possibility to reduce the derivation, all possible ways of reducing the derivation (called *reduction sequences*) get stuck in a loop. Prawitz proposed this to be the distinctive feature of Russell's paradox.

Tennant (1982) considered a wide range of examples and showed that all prominent mathematical and logical paradoxes follow this pattern. The steps playing the role of $(\in \mathrm{I})$ and $(\in \mathrm{E})$ are called *id est inferences*, as they result from extra-logical principles: In the case of Russell's paradox, from set-theoretic comprehension. In the case of the liar paradox, to take another example, analogous *id est* inferences would be based on the observation that a certain sentence says of itself that it is not true. Here, "observation" is not necessary empirical inspection, but may result from some arithmetical referencing mechanism.[1]

[1] Instead of looping reduction sequences one can, more generally, consider non-terminating reduction sequences, which covers paradoxes such as Yablo's (see Tennant, 1995). In the following, we will throughout speak of the *looping feature* of paradoxical derivations, keeping in mind that "non-termination" of reduction sequences is the appropriate more general term.

The Prawitz-Tennant analysis of paradoxes is a way to characterize paradoxes by their proof-theoretic behaviour, looking at the derivation of absurdity generated. Although this is not *per se* a solution to the paradoxes and Tennant stresses is should not be meant as such (see, e.g., Tennant, 1982, 268), it can be naturally turned into a solution (as implicitly suggested by both Prawitz and Tennant and, in a more explicit manner, by Schroeder-Heister, 2012b; Tranchini, 2015). Derivations in natural deduction aim at representing proofs. According to Prawitz and Tennant, however, only normalizable derivations 'really' represent proofs. Tennant's moral is thus the following:

> The general loss of normalisability [...] is a small price to pay for the protection it gives against paradox itself. Logic plays its role as an instrument of knowledge only insofar as it keeps proofs in sharp focus, through the lens of normality. (Tennant, 1982, 284)

As the paradoxical derivations of absurdity do not normalize, they are no 'real' proofs.

2 Ekman's paradox

Suppose we have derived A by means of a derivation $\mathcal{D}$. By assuming $A \to B$, $\mathcal{D}$ can be extended by $(\to \mathrm{E})$ to a derivation of conclusion B. By further assuming $B \to A$ one can conclude A by another application of $(\to \mathrm{E})$. Ekman (1994; 1998) observed that patterns of this kind, though not belonging to the official set of redexes, certainly constitute a redundancy, which can easily be removed as follows:

$$\dfrac{B \to A \qquad \dfrac{A \to B \qquad \begin{matrix}\mathcal{D}\\ A\end{matrix}}{B}\,(\to \mathrm{E})}{A}\,(\to \mathrm{E}) \qquad \rhd_E \qquad \begin{matrix}\mathcal{D}\\ A\end{matrix} \tag{3}$$

We call this step *Ekman's reduction* and the occurrence of B in the left derivation, which is removed by Ekman's reduction, an *Ekman-redundant* formula.

Consider now the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to \neg A \qquad \overset{1}{A}}{\neg A}\,(\to \mathrm{E}) \qquad \overset{1}{A}}{\bot}\,(\to \mathrm{E})}{\boxed{\neg A}}\,(\to \mathrm{I})(1) \qquad \dfrac{\neg A \to A \qquad \dfrac{\dfrac{\dfrac{A \to \neg A \qquad \overset{1}{A}}{\neg A}\,(\to \mathrm{E}) \qquad \overset{1}{A}}{\bot}\,(\to \mathrm{E})}{\neg A}\,(\to \mathrm{I})(1)}{A}\,(\to \mathrm{E})}{\bot}\,(\to \mathrm{E}) \tag{4}$$

Since the encircled occurrence of $\neg A$ is a redundant implication formula, this derivation is not normal. By applying implication reduction $\rhd_{\to}$ we obtain the following derivation:

$$\dfrac{A\to\neg A \qquad \dfrac{\neg A\to A \qquad \dfrac{\dfrac{\dfrac{A\to\neg A \quad \overset{1}{A}}{\neg A}\,(\to\mathrm{E}) \quad \overset{1}{A}}{\bot}\,(\to\mathrm{E})}{\neg A}\,(\to\mathrm{I})(1)}{\textcircled{A}}\,(\to\mathrm{E})}{\neg A}\,(\to\mathrm{E}) \qquad \dfrac{\neg A\to A \qquad \dfrac{\dfrac{\dfrac{A\to\neg A \quad \overset{1}{A}}{\neg A}\,(\to\mathrm{E}) \quad \overset{1}{A}}{\bot}\,(\to\mathrm{E})}{\neg A}\,(\to\mathrm{I})(1)}{A}\,(\to\mathrm{E})}{\bot}\,(\to\mathrm{E})$$

Here the encircled occurrence of A is an Ekman-redundant formula. By applying Ekman's reduction $\rhd_E$ we obtain the derivation (4) we started with. Since at each of the two steps there was only a single possibility to reduce the derivation, we are caught in a loop. This loop is not only a feature of the particular derivation (4). Ekman demonstrated that in the formulation of minimal logic considered there is no derivation of absurdity from $A\to\neg A$ and $\neg A\to A$, which is normal with respect to $\rhd_\to$ and $\rhd_E$.

The instance of Ekman's reduction (3) that is used in Ekman's derivation (4)

$$\dfrac{A\to\neg A \qquad \dfrac{\neg A\to A \quad \overset{\mathcal{D}}{\neg A}}{A}\,(\to\mathrm{E})}{\neg A}\,(\to\mathrm{E}) \qquad \rhd_E \qquad \overset{\mathcal{D}}{\neg A} \tag{5}$$

can be viewed as a translation of the corresponding instance of the $\in$-reduction (1) used in the derivation of Russell's paradox (2).

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\neg\lambda}}{\lambda}\,(\in\mathrm{I})}{\neg\lambda}\,(\in\mathrm{E}) \qquad \rhd_\in \qquad \overset{\mathcal{D}}{\neg\lambda} \tag{6}$$

The difference between (5) and (6) consists only in the fact that the redundancy is in one case generated by *id est* inferences, whereas in Ekman's case it is mimicked in propositional logic by applications of modus ponens. The major premises of modus ponens represents the rule applied in *id est* inferences. As Ekman puts it:

> Whatever motivation we have for [the instance (6) of $\in$-reduction $\rhd_\in$] this motivation also applies to [the instance (5) of Ekman's reduction $\rhd_E$] since the two reductions, from an informal point of view, are one and the same, but expressed using two different formal systems. (1994, 148; 1998, 78)

Given this observation, paradoxical derivations can be analysed as consisting of an extra-logical construction which is plugged into a portion of purely propositional reasoning. The extra-logical part is constituted by *id est* inferences which allows one to pass over, for some specific λ, from $\neg\lambda$ to λ and back. The logical part consists of the derivation (4) of absurdity $\bot$ from $\neg A \to A$ and $A \to \neg A$, for an unspecific (i.e. for all) A. Ekman's paradox would thus show that loops are not a feature of the extra-logical part, but of the logical part of paradoxical derivations. The looping feature would not depend on the possibility to move, for a certain λ, from λ to $\neg\lambda$ and vice versa, but that we can move, for any formula A, from $A \leftrightarrow \neg A$ to absurdity[2].

We do not take this to be the right conclusion to be drawn from the phenomenon observed by Ekman. Rather, we take Ekman's paradox to push the question of when a certain reduction counts as acceptable: Whether a derivation is normal depends on the collection of reductions adopted, and hence Tennant's criterion requires a particular attention in what should be taken to be a good reduction. In particular, Ekman's phenomenon shows that on a too loose notion of reduction, one obtains a too coarse criterion of paradoxality.

Before presenting substantial reasons for denying the goodness of Ekman's reduction, we would like to stress the fact that the problem of choosing the collection of reductions is crucial also to avoid the converse problem, namely of a too narrow criterion for paradoxicality.

3 Paradoxes and Classical Logic

Rogerson (2007) criticises Tennant's criterion in that it fails to detect a paradox when there is one. In particular she considers a formulation of Curry's paradox in classical logic and observes that the derivation fails to display the loopy feature called for by the Prawitz-Tennant analysis. We consider a slight variation of Rogerson's proof based on Russell's rather than Curry's paradox.

In the presence of the classical rule of *reductio ad absurdum*

$$\dfrac{\begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}\,(\mathrm{RAA})$$

[2]Ekman (1994) also investigated more general forms of reductions in propositional logic related to Crabbé's (1974) example (see Sundholm, 1979) of a non-normalizing derivation in set theory based on Zermelo's subset axiom rather than the unrestricted comprehension rules. This was one of the starting points of Hallnäs's (1983) work, on which Ekman's (1994) work, whose main subject is normalization in set theory, builds.

the derivation of Russell's paradox (2) can be recast in a more symmetric fashion:[3]

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}\,(\in \mathrm{E})\qquad \overset{1}{\lambda}}{\bot}\,(\to \mathrm{E})}{\boxed{\neg\lambda}}\,(\to \mathrm{I})(1) \qquad \dfrac{\dfrac{\overset{1}{\neg\lambda}\qquad \dfrac{\overset{1}{\neg\lambda}}{\lambda}\,(\in \mathrm{I})}{\bot}\,(\to \mathrm{E})}{\lambda}\,(\mathrm{RAA})\ (1)}{\bot}\,(\to \mathrm{E})$$

This derivation can be reduced by an application of implication reduction $\rhd_{\to}$ to the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\neg\lambda}\qquad \dfrac{\overset{1}{\neg\lambda}}{\lambda}\,(\in \mathrm{I})}{\bot}\,(\to \mathrm{E})}{\lambda}\,(\mathrm{RAA})\ (1)}{\neg\lambda}\,(\in \mathrm{E}) \qquad \dfrac{\dfrac{\overset{1}{\neg\lambda}\qquad \dfrac{\overset{1}{\neg\lambda}}{\lambda}\,(\in \mathrm{I})}{\bot}\,(\to \mathrm{E})}{\lambda}\,(\mathrm{RAA})\ (1)}{\bot}\,(\to \mathrm{E})$$

According to Rogerson, this derivation cannot be further reduced:[4]

> No standard reduction steps given by Prawitz in 1965 straightforwardly apply in this case as the use of the λ operator insulates the formulas from the normalization process. It seems plausible to conclude that this proof does not reduce to a normal form and does not generate a non-terminating reduction sequence in the sense of Tennant (1982) or Tennant (1995). Thus, Tennants criterion for paradoxicality does not apply here. (This is not to say that it is inconceivable that someone might be able to define a reduction step applicable in this case that would induce a non-terminating reduction sequence.)

Although it is true that no standard reduction step given by Prawitz applies to this derivation, it is also well known that the normalization strategy for classical logic

[3]By 'symmetric' we mean that the two main sub-derivations can be obtained from each other by replacing occurrences of λ with occurrences of $\neg\lambda$ (and vice versa) and by switching the order of the premises of $(\to \mathrm{E})$.

[4]Although Rogerson speaks of a derivation based on Curry's paradox, the derivation we discuss can be viewed as obtained from the last derivation on p. 174 of Rogerson (2007) by replacing $a \in a$ with λ and p with $\bot$, and moreover by (i) removing in both main sub-derivations redundant applications of (RAA), i.e. applications allowing one to pass from $\bot$ to $\bot$ with no discharge; (ii) replacing in both sub-derivations the pattern $\dfrac{\dfrac{\neg\lambda \quad \overset{1}{\lambda}}{\bot}}{\neg\lambda}\,(1)$ with just $\neg\lambda$. The considerations in this section apply exactly to Rogerson's original derivation as well.

devised by Prawitz (1965) applies only to language fragments for which the application of (RAA) can be restricted to atomic formulas. In richer languages, for example in languages containing disjunction and existential quantification, the conclusion of (RAA) cannot be restricted without loss of generality to atomic formulas, and in order for normal derivation to enjoy the sub-formula property a further (family of) reduction(s) has to be considered. This new reduction is based on the idea that the conclusion of an application of (RAA) which is also the major premise of an elimination rule counts as a redundancy to be eliminated. The reduction, proposed by Stålmarck (1991), can be depicted schematically as follows:

$$
\dfrac{\dfrac{\begin{matrix}[\neg \overset{n}{A}] \\ \mathcal{D} \\ \bot\end{matrix}}{A}\,(\mathrm{RAA})\ (n) \qquad \overline{\mathcal{D}}}{B}\,(\dagger\mathrm{E})
\qquad \rhd_{\mathrm{RAA}} \qquad
\begin{matrix}
\dfrac{\dfrac{\overset{n}{\neg B} \qquad \dfrac{\overset{m}{A} \quad \overline{\mathcal{D}}}{B}\,(\dagger\mathrm{E})}{\bot}}{[\neg A]}\,(\rightarrow\mathrm{I})(m) \\ \mathcal{D} \\ \dfrac{\bot}{B}\,(\mathrm{RAA})\ (n)
\end{matrix}
$$

where (†E) stands for an application of an elimination rule for some connective † belonging to the language fragment under consideration and $\overline{\mathcal{D}}$ stands for the (possibly empty) list of derivations of the minor premises of the application of (†E).

In the language of naive set theory, the presence of the operators for the formation of terms for sets jeopardizes the notion of atomic sentence. Thus, redundant conclusions of (RAA) are not just non-atomic formulas, but more generally any formula which can act as major premise of an elimination rule. This makes plausible the idea of letting †E in the scheme for reduction proposed by Stålmarck to range over ($\in$ I) as well. Once this is done, Rogerson's derivation can be further reduced, and the derivation after some steps reduces back to itself[5].

Tennant formulated his criterion for paradoxality with an emphasis on intuitionistic logic, by claiming that a paradoxical sentence is one governed by *id est* inferences such that there are derivations of $\bot$ in the extension of intuitionistic logic with these inferences that fail to normalize. As observed by Rogerson, the choice of intuitionistic logic is certainly motivated by the will of showing that non-constructive principles of reasoning do not play any significant role in the phenomenon of paradoxes. However, part of the reason for this choice is also the fact that the criterion for paradoxality is formulated in terms of normalization, and intuitionistic logic (in its usual formulation at least) is well-behaved with respect to normalization. Given the crucial role played by normalization (not only from the formal, but also from the conceptual point of view, as stressed by Tennant in the quote at the end of section 1), the "base" system relative

[5]Provided that, as usual, one also reduces redundant applications of (RAA) (see footnote 4 above). Otherwise, one ends up with the more general kind of non-termination mentioned in footnote 1.

to which the non-normalizability effects of *id est* inference is to be tested must enjoy normalization.

Tennant may be wrong in restricting the attention to intuitionistic logic, but we do not believe that extending the criterion beyond this logic is as problematic as Rogerson claims. For the case of classical logic, the above observations are sufficient to show that on a proper account of normalization for classical logic, Russell's (and Curry's paradox as well) display the looping effect called for by Prawitz's and Tennant's analysis. Rogerson hints at other possible counterexamples, but, provided the logical frameworks in the background can be given a proper proof-theoretic presentation, Tennant's criterion should always be applicable. In fact, if any derivation in the "base" system reduces to a normal one, and if normal derivations enjoy the sub-formula property, then the derivation of absurdity in the system extended with the *id est* inferences cannot normalize. This holds at least when the *id est* inferences display a certain symmetry, for instance by being introduction and elimination rules for a certain expression equipped with a reduction to get rid of the redundancies constituted by consecutive applications of introduction and elimination rules (for a discussion see Tranchini, to appear).

4 Paradoxes and Reduction

We now turn back to the original question posed at the end of Section 2, namely whether the reduction proposed by Ekman should be taken for good. In almost all presentations of natural deduction, reductions are presented as means to get rid of redundancies within proofs. This is also the background of Tennant's analysis, who writes (Tennant, 1995, 199–200):

> The reduction procedures for the logical operators are designed to eliminate such unnecessary detours within proofs.
>
> So are other abbreviatory procedures σ, which have the general form of 'shrinking' to a single occurrence of A, any logically circular segments of branches (within the proof) of the form shown below to the left:
>
> $$\begin{array}{c} \dfrac{\dfrac{A}{B_1}}{\vdots} \\ \dfrac{B_n}{A} \end{array} \quad \rhd_\sigma \quad A$$
>
> One thereby identifies the top occurrence of A with the bottom occurrence of A, and gets rid of the intervening occurrences of $B_1, \ldots, B_n$, that form the filling of this unwanted sandwich. Logically, one should live by bread alone.

Given this, Tennant should have nothing to object against the reduction $\rhd_E$ proposed by Ekman as it is a variant of $\rhd_\sigma$. However, the understanding of reduction as "abbreviatory procedures" is not the only possible one. We actually claim that this understanding is not appropriate for meaning-theoretical investigations, and take Ekman's paradox to be a striking phenomenon that points to this fact.

From a semantical standpoint[6] proofs may be viewed as abstract entities linguistically represented by natural deduction derivations. Reduction procedures for derivations can then be viewed as yielding a criterion of identity between proofs (Prawitz, 1971, § I.3.5.6), in the following sense: The reflexive, symmetric and transitive closure of the relation of one-step reducibility clearly induces an equivalence relation between the derivations of a logical system, and derivations equivalent in this sense are considered to represent the same proof. For what follows only this identity criterion is relevant, not the exact nature of 'proofs' beyond this. For simplicity proofs might, for example, be considered to be equivalence classes with respect to the equivalence of derivations.

A typical example of two derivations of the same formula which represent two distinct proofs of it is this:

$$\dfrac{\dfrac{\overset{1}{A}}{A \to A}\,(\to\mathrm{I})(1)}{A \to (A \to A)}\,(\to\mathrm{I}) \qquad\qquad \dfrac{\dfrac{\overset{1}{A}}{A \to A}\,(\to\mathrm{I})}{A \to (A \to A)}\,(\to\mathrm{I})(1)$$

In the two derivations the assumption is discharged at different places, which is intuitively a reason to consider them to be two ways of establishing the formula $A \to (A \to A)$.

On the standard reductions, the two derivations in fact do belong to two different equivalence classes induced by the reflexive, symmetric and transitive closure of reducibility. This is a consequence of the fact that both are in normal form and of a property of the reduction system based on implication reduction, namely the Church-Rosser property (also called 'confluence'). According to it, if one derivation reduces (in a number of steps) to two distinct derivations, then there should be a third one to which the latter two both reduce (in some finite numbers of steps). An immediate consequence of this is that two different derivations in normal form (thus two derivations which do not reduce further to the same derivation) can never be obtained by reducing some derivation. In other words, there is no chain of reductions which connects the two derivations, thus they belong to different equivalence classes.

The above example thus shows that identity of proofs is not trivial (given the standard reductions). A natural requirement for the addition of a new reduction could be that of not trivializing identity of proof, in the sense that is should always be possible

[6] In in the sense of 'proof-theoretic semantics', see Schroeder-Heister (2012a)

to exhibit two derivations of the same conclusion belonging to two distinct equivalence classes.

It is actually known that any reduction extending the equivalence relation induced by the reduction for implication and the following expansion:

$$\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \quad \rhd_{\to\text{-exp}} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \quad \overset{1}{A}}{B}\,(\to\mathrm{E})}{A \to B}\,(\to\mathrm{I})(1)$$

trivializes the identity of proofs in the implicational fragment of intuitionistic logic (Došen & Petrić, 2001; Widebäck, 2001).[7]

However, we will show that Ekman's reduction is sufficient to trivialize the identity of proofs induced by implication reduction alone. On such an understanding, the notion of reduction is much narrower than the one arising from taking reductions as 'abbreviation procedures". On this narrower conception Ekman's alleged reduction turns out to be no reduction at all.

Although the notion of identity of proof is quite established in general and categorial proof theory (see Došen, 1997, 2000, 2003; Došen & Petrić, 2004) as well as in investigation on the lambda calculus, it is usually disregarded in presentations of natural deduction. We therefore discuss at some length a simple counterexample vindicating Ekman's reduction as inappropriate, with the hope of stressing the relevance of these formal results for philosophically informed proof-theory.

To begin with, instead of (3) we actually consider Ekman's reduction $\rhd_E$ in the more general form

$$\dfrac{\begin{array}{c}\mathcal{D}''\\ B \to A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}'\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}\\ A\end{array}}{B}\,(\to\mathrm{E})}{A}\,(\to\mathrm{E}) \qquad \rhd^{*}_{E} \qquad \begin{array}{c}\mathcal{D}\\ A\end{array} \tag{7}$$

i.e., we allow for $A \to B$ and $B \to A$ to be obtained by derivations $\mathcal{D}'$ and $\mathcal{D}''$. This means that we assume that Ekman's reduction is closed under substitution of derivations for open assumptions. In section 5 we show that even without this generalization a corresponding counterexample can be given.

For simplicity of exposition, we consider a language containing conjunction. A corresponding, but less well readable example could be given in the implicational language considered so far. The rules and reductions associated to conjunction are the following:

$$\dfrac{A \quad B}{A \wedge B}\,(\wedge\mathrm{I}) \qquad\qquad \dfrac{A \wedge B}{A}\,(\wedge\mathrm{E1}) \qquad \dfrac{A \wedge B}{B}\,(\wedge\mathrm{E2})$$

[7] This result is what in the typed lambda calculus corresponds to a corollary of Böhm's theorem for the untyped lambda calculus.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \wedge B}(\wedge\mathrm{I})}{A}(\wedge\mathrm{E1}) \quad \rhd_{\wedge 1} \quad \begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \wedge B}(\wedge\mathrm{I})}{B}(\wedge\mathrm{E2}) \quad \rhd_{\wedge 2} \quad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}$$

Consider now the the formulas $A \wedge A$ and A and the following proofs of their mutual implications:[8]

$$\dfrac{\dfrac{\overset{1}{A \wedge A}}{A}(\wedge\mathrm{E1})}{(A \wedge A) \to A}(\to\mathrm{I})(1) \qquad\qquad \dfrac{\dfrac{\overset{1}{A} \quad \overset{1}{A}}{A \wedge A}(\wedge\mathrm{I})}{A \to (A \wedge A)}(\to\mathrm{I})(1)$$

Given an arbitrary derivation $\mathcal{D}$ of $A \wedge A$, consider the following derivation $\mathcal{D}'$:

$$\mathcal{D}' = \left\{ \dfrac{\dfrac{\dfrac{\overset{1}{A} \quad \overset{1}{A}}{A \wedge A}(\wedge\mathrm{I})}{A \to (A \wedge A)}(\to\mathrm{I})\ (1) \qquad \dfrac{\dfrac{\dfrac{\overset{2}{A \wedge A}}{A}(\wedge\mathrm{E1})}{(A \wedge A) \to A}(\to\mathrm{I})(2) \quad \begin{matrix}\mathcal{D} \\ A \wedge A\end{matrix}}{A}(\to\mathrm{E})}{A \wedge A}(\to\mathrm{E}) \right.$$

Ekman's (starred) reduction enables the performance of the following reduction step:

$$\begin{matrix}\mathcal{D}' \\ A \wedge A\end{matrix} \quad \rhd \quad \begin{matrix}\mathcal{D} \\ A \wedge A\end{matrix} \tag{8}$$

Since in $\mathcal{D}'$ both $(A \wedge A) \to A$ and $A \to (A \wedge A)$ are maximum formulas, we also have the following reduction, which is obtained by applying $\rhd_{\to}$ twice:

$$\begin{matrix}\mathcal{D}' \\ A \wedge A\end{matrix} \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ A \wedge A\end{matrix}}{A}(\wedge\mathrm{E1}) \quad \dfrac{\begin{matrix}\mathcal{D} \\ A \wedge A\end{matrix}}{A}(\wedge\mathrm{E1})}{A \wedge A} \tag{9}$$

Now suppose that $\mathcal{D}$ has the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{A \wedge A}(\wedge\mathrm{I})$$

for some arbitrary derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of A.

Then (8) and (9) give us the reductions

$$\begin{matrix}\mathcal{D}' \\ A \wedge A\end{matrix} \quad \rhd \quad \dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ A\end{matrix}}{A \wedge A}(\wedge\mathrm{I}) \tag{10}$$

---

[8] A substantially equivalent counterexample in the implicational fragment can be obtained using the formulas $(A \to (A \to B)$ and $A \to B$.

and

$$\begin{array}{c}\mathcal{D}' \\ A\wedge A\end{array} \quad \rhd \quad \dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & A\end{array}}{A \wedge A}\,(\wedge\mathrm{I})}{A}\,(\wedge\mathrm{E1}) \qquad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & A\end{array}}{A \wedge A}\,(\wedge\mathrm{I})}{A}\,(\wedge\mathrm{E1})}{A\wedge A}\,(\wedge\mathrm{I}) \quad \rhd_{\wedge 1} \quad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_1 \\ A & A\end{array}}{A \wedge A}\,(\wedge\mathrm{I}) \tag{11}$$

respectively. Therefore the adoption of Ekman's (starred) reduction implies that the following two derivations

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_1 \\ A & A\end{array}}{A \wedge A} \qquad\qquad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & A\end{array}}{A \wedge A}$$

are equivalent with respect to reducibility, i.e. that they represent the same proof. This means that also the following two derivations, which result from the previous ones by extending each of them with an application of ($\wedge$E2):

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_1 \\ A & A\end{array}}{A \wedge A}}{A}\,\wedge\mathrm{E2} \qquad\qquad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ A & A\end{array}}{A \wedge A}}{A}\,\wedge\mathrm{E2}$$

are equivalent with respect to reducibility. If we apply to each of them the reduction $\rhd_{\wedge 2}$, we obtain obtain the two derivations

$$\begin{array}{c}\mathcal{D}_1 \\ A\end{array} \qquad\qquad \begin{array}{c}\mathcal{D}_2 \\ A\end{array}$$

meaning that $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent with respect to reducibility. Therefore by using Ekman's (starred) reduction in addition to the standard reductions, we can show that any two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of a formula A represent the same proof. If every proof of A is represented by a derivation of A, there cannot be two different proofs of A: Any two proofs of a provable formula A are identical. This is a devastating consequence, if we are interested not just in provability, but in the structure of proofs. If we require that reductions do not trivialize the notion of identity of proofs, Ekman's transformation does not count as a reduction.

We thus propose to amend Tennant's paradoxality criterion by requiring that reductions do not trivialize identity of proofs. In this way the problem posed by Ekman's result for the Prawitz-Tennant test for paradoxality is resolved in that Ekman's derivation (4) now fails to count as a paradox.

5 The Issue of Substitution

In defence of Ekman, one might argue that he formulates his reduction with $A \to B$ and $B \to A$ in assumption position according to (3), whereas to show that his reduction trivializes identity of proofs we considered the generalized form (7). This generalized form is closed under substitution of derivations for open assumptions. Now it is hard to make sense of a notion of reduction *not* closed under substitution in this sense. However, the following example demonstrates our trivialization result even without this assumption, on the basis of Ekman's reduction in the form (3).

The following derivation (encircled is an Ekman redundant formula)

$$
\begin{prooftree}
\AxiomC{$\overset{1}{A \to (A \wedge A)}$}
\AxiomC{$\overset{2}{(A \wedge A) \to A}$}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_2$}
\noLine
\UnaryInfC{A}
\RightLabel{$(\wedge\mathrm{I})$}
\BinaryInfC{$A \wedge A$}
\RightLabel{$(\to\mathrm{E})$}
\BinaryInfC{$\textcircled{$A$}$}
\RightLabel{$(\to\mathrm{E})$}
\BinaryInfC{$A \wedge A$}
\RightLabel{$(\to\mathrm{I})\ (1)$}
\UnaryInfC{$(A \to (A \wedge A)) \to (A \wedge A)$}
\RightLabel{$(\to\mathrm{I})\ (2)$}
\UnaryInfC{$((A \wedge A) \to A) \to (A \to (A \wedge A)) \to (A \wedge A))$}
\AxiomC{$\overset{3}{A \wedge A}$}
\RightLabel{$(\wedge\mathrm{E1})$}
\UnaryInfC{A}
\RightLabel{$(\to\mathrm{I})(3)$}
\UnaryInfC{$(A \wedge A) \to A$}
\RightLabel{$(\to\mathrm{E})$}
\BinaryInfC{$(A \to (A \wedge A)) \to (A \wedge A)$}
\AxiomC{$\overset{4}{A}$}
\AxiomC{$\overset{4}{A}$}
\RightLabel{$(\wedge\mathrm{I})$}
\BinaryInfC{$A \wedge A$}
\RightLabel{$(\to\mathrm{I})\ (4)$}
\UnaryInfC{$A \to (A \wedge A)$}
\RightLabel{$(\to\mathrm{E})$}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
$$

reduces via Ekman's reduction $\rhd_E$ to the following (in which the applications of $(\to\mathrm{I})$ without numeral do not discharge anything):

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_2$}
\noLine
\UnaryInfC{A}
\BinaryInfC{$A \wedge A$}
\RightLabel{$(\to\mathrm{I})$}
\UnaryInfC{$(A \to (A \wedge A)) \to (A \wedge A)$}
\RightLabel{$(\to\mathrm{I})$}
\UnaryInfC{$((A \wedge A) \to A) \to (A \to (A \wedge A)) \to (A \wedge A))$}
\AxiomC{$\overset{3}{A \wedge A}$}
\RightLabel{$\wedge\mathrm{E1}$}
\UnaryInfC{A}
\RightLabel{$(\to\mathrm{I})(3)$}
\UnaryInfC{$(A \wedge A) \to A$}
\BinaryInfC{$(A \to (A \wedge A)) \to (A \wedge A)$}
\AxiomC{$\overset{4}{A}$}
\AxiomC{$\overset{4}{A}$}
\BinaryInfC{$A \wedge A$}
\RightLabel{$(\to\mathrm{I})\ (4)$}
\UnaryInfC{$A \to (A \wedge A)$}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
$$

which in turn reduces via two applications of $\rhd_{\to}$ to

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_2$}
\noLine
\UnaryInfC{A}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
$$

On the other hand, by applying first $\rhd_{\to}$ (for four times) and then $\rhd_{\wedge 1}$ (twice) to the first derivation one obtains

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
$$

In other words, we have that the two derivations

$$
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_2$}
\noLine
\UnaryInfC{A}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
\qquad\qquad
\begin{prooftree}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\AxiomC{$\mathcal{D}_1$}
\noLine
\UnaryInfC{A}
\BinaryInfC{$A \wedge A$}
\end{prooftree}
$$

are equivalent with respect to reducibility even when one adopts the restricted form of Ekman's reduction. Thus the restricted form of Ekman's reduction is sufficient to trivialize identity of proofs (by the argument given at the end of the previous section).

Outlook

Ekman's 'paradox' not only teaches us the importance of an appropriate notion of reduction for formulating a proof-theoretic criterion of paradoxality, but also tells us something about the nature of paradoxical sentences. What triggers a genuine paradox is not simply the assumption that a sentence is *interderivable* with its own negation, as in Ekman's derivation (4). A genuine paradox is a sentence A such that there are proofs from A to $\neg A$ and from $\neg A$ to A that composed with each other give us the identity proof A (i.e., the formula A considered a proof of A from A). Such a notion, which is stricter than just interderivability, and which is well-known in general (in particular categorial) proof theory as *isomorphism* of formulas (see Došen (2006)), must be given a much more prominent role in proof-theoretic semantics than it currently enjoys.

Acknowledgements The first author is deeply indebted to Lars Hallnäs, who in the early 1990s draw his attention to Ekman's results and pointed out to him their conceptual significance. The final version of this paper profited heavily from extensive discussions of both authors with Kosta Došen, Jean-Baptiste Joinet, Göran Sundholm and Zoran Petrić. We are also grateful to two anonymous reviewers for their helpful comments and suggestions. This work was carried out within the French-German ANR-DFG projects "Hypothetical Reasoning — Its Proof-Theoretic Analysis" (DFG Schr 275/16-2) and "Beyond Logic" (Schr 275/17-1), and the DFG project "Logical Consequence — Proof-Theoretical and Epistemological Perspectives" (DFG Tr 1112/1-1).

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